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Time evolution of the distribution function of fast neutral beam injected into a plasma

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Abstract : We have studied in this paper, the time evolution of a neutral beam injected into a plasma. The relaxation time of neutral beam having initially a δ-function shaped distribution function, can be estimated by calculating the eigenvalues of Fokker-Plank Operator (FPO) The coefficients of the FPO for our problem are determined from Rosenbluth potentials. Stationary and nonstationary solutions are found and discussed for the distribution function of neutral test particles

Keywords : Relaxation time, neutral beam injection, Rosenbluth potentials

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1. Introduction

The relaxation of charged species dispersing in a fully ionized plasma has been studied by Shizgal [1,2]. The method used by this author, is based on the expansion of the distribution function in the eigenfunctions of Fokker-Plank operator. Reciprocals of the corresponding eigenvalues give the relaxation times of the system. We employ this method to study the relaxation of a fast neutral beam injected into an homogeneous plasma for heating purposes (NBI). We suppose that a beam of energetic neutral particles which will be considered as our 'test particles' enter an homogeneous plasma of the same species. The neutral beam are supposed here to be neutral hydrogen atoms injected into an hydrogen plasma. The distribution function of test particles could be taken approximately as a δ-function, while that of plasma particles is considered Maxwellian and spatially isotropic. In Section 2, we will write Fokker-Plank (FP) equation for NBI problem and determine its coefficients by calculation of Rosenbluth potentials. In Section 3, we will discuss stationary and nonstationary solutions of FP equation. Finally, time evolution of an initially δ-function shaped distribution function of test particles will be given qualitatively.

2. Fokker-Plank equation for NBI problem

Obviously, we can express the time evolution of distribution functions of different interacting particles by means of Fokker-Plank equation. We represent respectively by f_t and f_b the distribution function of test and plasma particles. Thus the evolution of test particle will be given by FP equation of the form :

$$\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial v}\left[A_t f_t - \frac{1}{2}\frac{\partial}{\partial v}(D_t f_t)\right]. \tag{1}$$

For an isotropic distribution, the coefficients A_t and D_t are defined as [3,4] :

$$A_s^t = -v_s^t(v)\mathbf{v}, \tag{2}$$

$$D_t(v) = D_\perp^t(v)\left[1 - \frac{\mathbf{v}\mathbf{v}}{v^2}\right] + D_\parallel^t(v)\frac{\mathbf{v}\mathbf{v}}{v^2}, \tag{3}$$

where
$$v_s^t(v) = -\Gamma_t \sum_b Z_b^2\left(1 + \frac{m_t}{m_b}\right)\frac{1}{v}\frac{dh_b}{dv}, \tag{4}$$

$$D_\perp^t(v) = \frac{\Gamma_t}{v}\sum_b Z_b^2 \frac{dg_b}{dv}, \tag{5}$$

and
$$D_\parallel^t = \Gamma_t \sum_b Z_b^2 \frac{d^2 g_b}{dv^2}. \tag{6}$$

The summation over b includes all plasma constituent particles (electrons and ions) g_b and h_B are the famous Rosenbluth potentials. The coefficient Γ_t in eqs. (4), (5) and (6) is given by $\Gamma_t = \dfrac{4\pi Z_t^2 e^4}{m_t}(\ln \Lambda)$, where $\ln\Lambda$ is the coulomb logarithm. Rosenbluth potentials are defined as [5]

$$g_b(\mathbf{v}) = \int d^3v' f_b(v')\left|\mathbf{v} - \mathbf{v}'\right|, \tag{7}$$

$$h_b(\mathbf{v}) = \int d^3v' \frac{f_b(\mathbf{v}')}{\left|\mathbf{v} - \mathbf{v}'\right|}, \tag{8}$$

where f_b is the distribution function for plasma particles and is considered as mentioned earlier to be Maxwellian, *i.e.*, $f_b(v) = \dfrac{n_b}{\pi^{3/2} v_b^2}\exp\left(-\dfrac{v^2}{v_b^2}\right)$. Here, $v_b = \left[\dfrac{2T_b}{m_b}\right]^{1/2}$.
Substituting f_b into eqs. (7) and (8), we obtain after carrying the required integrations :

$$h_b(v) = \frac{n_b \Phi\left(\frac{v}{v_b}\right)}{v}, \tag{9}$$

$$g_b(v) = n_b v_b \phi\left(\frac{v}{v_b}\right), \tag{10}$$

where $$\phi(x) = \left(x + \frac{1}{2x}\right)\Phi(x) + \frac{1}{\sqrt{\pi}} e^{-x^2} \tag{11}$$

and $\Phi(x)$ is the error function. Using these and eqs. (4–6), we get

$$\nu_s^t(v) = \frac{2\Gamma_t}{v} \sum_b \frac{n_b Z_b^2}{v_b^2}\left(1 + \frac{m_t}{m_b}\right) G\left(\frac{v}{v_b}\right), \tag{12}$$

where $$G(x) = \frac{\phi(x) - x\phi'(x)}{2x^2}. \tag{13}$$

We now expand eq. (12) for plasma electrons and ions as

$$\nu_t^s(v) = \frac{2\Gamma_t}{v}\left[\frac{n_e Z_e^2}{v_e^2}\left(1+\frac{m_t}{m_b}\right)G\left(\frac{v}{v_e}\right) + \frac{n_i Z_i^2}{v_i^2}\left(1+\frac{m_t}{m_i}\right)G\left(\frac{v}{v_i}\right)\right]. \tag{14}$$

We assume further that the relatively high velocity of test particles (v) lies between plasma ions and electrons velocities, *i.e.*, $v_i << v << v_e$. The following assymptotic limits thus will be obtained for $\phi(x)$ and $G(x)$:

$$X \to 0, \qquad \phi(x) = \frac{2\pi}{\sqrt{\pi}}, \qquad G(x) = \frac{2x}{3\sqrt{\pi}}; \tag{15}$$

$$X \to \infty, \qquad \phi(x) = 1, \qquad G(x) = \frac{1}{2x^2}. \tag{16}$$

The slowing down rate will therefore be written in the simpler form of

$$\nu_s^t(v) = n_e\Gamma_t\left[\frac{4}{3\sqrt{\pi}}\left(1 + \frac{m_t}{m_e}\right)\frac{1}{v_e^3} + \frac{Z_i}{1}\left(1 + \frac{m_t}{m_e}\right)\frac{1}{v^3}\right], \tag{17}$$

which gives for the first coefficient of Fokker-Plank equation as

$$A_t(v) = -\left[\alpha v + \frac{\beta}{v^2}\right]. \tag{18}$$

Here, both test and plasma particles are supposed to be hydrogen. ($m_t \sim m_i >> m_e$)

and $$\alpha = -\frac{4n_e\Gamma_t m_t}{3\sqrt{\pi}\, m_e v_e^3}, \qquad \beta = -n_e\Gamma_t Z_i\left(1 + \frac{m_t}{m_i}\right) \approx -2n_e\Gamma_t Z_i. \tag{19}$$

The second coefficient of FP equation is obtained from eqs. (11) and (13) by carrying out the summation for two plasma species as

$$D_\perp^t(v) = \frac{\Gamma_t}{v}\left\{n_e Z_e^2\left[f\left(\frac{v}{v_e}\right) - G\left(\frac{v}{v_e}\right)\right] + n_i Z_i^2\left[\phi\left(\frac{v}{v_i}\right) - G\left(\frac{v}{v_i}\right)\right]\right\}. \tag{20}$$

Again, using the limit conditions, we find for the parallel and perpendicular (with respect to the beam) coefficients the following formulae :

$$D_{\parallel}^{t}(v) = \frac{n_e \Gamma_t}{v_e}\left[\frac{4}{3\sqrt{\pi}} + Z_i \frac{v_e v_i^2}{v^3}\right], \tag{21}$$

$$D_{\perp}^{t}(v) = n_e \Gamma_t\left[\frac{2Ze^2}{\sqrt{\pi}\, v_e} - \frac{2Ze^2}{3\sqrt{\pi}\, v_e} + \frac{Z_i}{v} - \frac{Z_i^2 v_i^2}{2v^3}\right]. \tag{22}$$

In the limit where $\frac{v_i}{v} \to 0$, the last term in eq. (22) could be eliminated, this leads to

$$D_{\perp}^{t}(v) = n_e \Gamma_t\left[\frac{4}{3\sqrt{\pi}\, v_e} + \frac{Z_i}{v}\right] \approx \frac{n_e \Gamma_t Z_i}{v}. \tag{23}$$

Comparing eqs. (21) and (23) we see that $D_{\parallel}^{t}$ is much smaller than $D_{\perp}^{t}(D_{\parallel}^{t} << D_{\perp}^{t})$. The diffusion coefficient will have therefore only the term perpendicular to the beam propagation. Substituting the coefficients $A = \alpha x + \frac{\beta}{x^2}$ and $B = \frac{\gamma}{x}$ into FP equation we get

$$\frac{\partial f_t}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\left(\frac{\gamma}{x}\right) f_t\right] - \frac{\partial}{\partial x}\left[\left(\alpha x + \frac{\beta}{x^2}\right) f_t\right]. \tag{24}$$

Thus the FP operator, $\frac{\partial f_t}{\partial t} = L_{FP} f_t$ can be recluced to the following form

$$L_{FP} = \frac{\partial^2}{\partial x^2}\left[\left(\frac{\gamma}{x}\right)\cdots\right] - \frac{\partial}{\partial x}\left[\alpha x + \frac{\beta}{x^2} + \cdots\right]. \tag{25}$$

3. Fokker-Plank equation's solutions

For solving FP equation (25) we have to find the eigenvalues and the eigenfunctions of FP operator. Reciprocals of these eigenvalues will then give the relaxation times of initial fast particles.

3.1. Stationary solution :

We first look for a stationary solution of Fokker-Plank equation. We designate this by $f(x, t) = f(x, \infty) = f_0(x)$. We have from eq. (25) for the stationary case :

$$\frac{\partial[A(x) f(x, \infty)]}{\partial x} + \frac{\partial^2[B(x) f(x, \infty)]}{\partial x^2} = 0. \tag{26}$$

Integrating this from 0 to x and rearranging it by introducing the normalization factor N, we get

$$f_0(x) = N \exp\left[-\int_0^x \frac{A(x') dx'}{B(x')} - \ln B(x)\right] = f(x, \infty).$$

For our purpose, we use $A(x) = -\left(\alpha x + \frac{\beta}{x^2}\right)$ and $B(x) = \frac{\gamma}{x}$. We get for the stationary solution

$$f_0(x) = NX^{-3} \exp\left(\frac{\alpha}{3\gamma} x^3\right),$$

or
$$f_0(v) = Nv^{-3} \exp\left(\frac{\alpha}{3\gamma} v^3\right). \tag{27}$$

3.2. Nonstationary solution :

In this section, we look for the solution which represents the time evolution of the distribution function of test particles (f_t). The starting point will be the following equation

$$Lf_t = -\lambda_n f_t, \tag{28}$$

where
$$L = \frac{d^2}{dx^2}\left[\frac{\gamma}{x} \cdots\right] - \frac{d}{dx}\left[\left(\alpha x + \frac{\beta}{x^2}\right) + \cdots\right], \tag{29}$$

and λ_n are corresponding eigenvalues. Reciprocals of these eigenvalues will give us the relaxation times of test particles. To proceed the problem, we take the function $f_t(x, t)$ as

$$f_t = f_n(x) = f \cdot \varphi_n(x).$$

So we have

$$Lf.\varphi_n + \lambda_n f_o \varphi_n = 0.$$

Substituting the terms corresponding to the operator, we get after some calculations and arranging in order of ϕ's derivatives

$$\frac{\gamma}{x} f_1 \phi_n'' + \left[\frac{-8\gamma - b}{x^2} + \alpha x\right] f_0 \phi_n' + \left[\frac{20\gamma + 5\beta}{x^3} - 4\alpha - \frac{\alpha\beta}{\gamma} + \lambda_n\right] f_0 \phi_n = 0. \tag{30}$$

From eq. (19) and $\gamma = \frac{n_e Z_i \Gamma_t}{2}$ the ratio $\frac{\beta}{\gamma}$ is found to be equal to -4. Dividing eq (30) by γ, we get after changing the variable $y = x^3$ the following equation

$$y\phi_n'' + \left[-\frac{2}{3} + \frac{\alpha}{3\gamma} y\right]\phi_n' + \frac{\lambda_n}{9\gamma}\phi_n = 0. \tag{31}$$

The next step is to use the following change of variables

$$\phi_n(y) = y^A \psi_n(y). \tag{32}$$

If we now substitute this into eq. (31), we get,

$$y\psi_n'' + \left[2A - \frac{2}{3} + \frac{\alpha}{3\gamma}y\right]\psi_n' + \left[\frac{A(A-1) - \frac{2A}{3}}{y} + \left(\frac{\lambda_n}{9\gamma} + \frac{\alpha A}{3\gamma}\right)\right]\psi_n = 0. \tag{33}$$

This will be similar to associated Laguerre equation if the first term in the last paranthesis vanishes. So

$$A = 0, \quad A = \frac{5}{3}.$$

Using $A = \frac{5}{3}$ in eq. (33) we obtain

$$y\psi_n'' + \left[\frac{8}{3} + \frac{\alpha}{3\gamma}y\right]\psi_n' + \left[\frac{\lambda_n}{9\gamma} + \frac{5\alpha}{9\gamma}\right]\psi_n = 0. \tag{34}$$

But associated Laguerre equation as defined in mathematics texts is of the form :

$$xL_n^{k''}(x) + (k+1-x)L_n^{k'}(x) + L_n^k(x) = 0. \tag{35}$$

Comparing these two equations and reminding that α is intrinsically negative (see eq. 19), eq. (34) takes the following form

$$x'\psi_n'' + \left[\frac{8}{3} - x'\right]\psi_n' + \left[\frac{5\alpha + \lambda_n}{3|\alpha|}\right]\psi_n = 0. \tag{36}$$

Here, we have used a new variable change, $\left|\frac{\alpha}{3\gamma}\right|y = x'$. Comparing eqs. (35) and (36), we obtain finally $K = \frac{5}{3}$ and $\gamma_n = 3n|\alpha| - 5\alpha$.

Eigenfunctions of FP operators are thus obtained using different values for n. Relaxation times will be given by

$$\tau_{\text{relax}} = \lambda_n^{-1} = \frac{1}{3n|\alpha| - 5\alpha}. \tag{37}$$

The expanded form of the associated Laguerre polynomial for our case $\left(K = \frac{5}{3}\right)$, is

$$L_n^{5/3}(x) = \sum_{m=0}^{n} (-1)^m \frac{\left(n + \frac{5}{3}\right)!}{(n-m)!\left(\frac{5}{3} + m\right)!\,m!} x^n. \tag{38}$$

The general form of the distribution function is thus obtained as

$$f_t(x,t) = (x^3)^{5/3} e^{-x^3} \sum_{n=0}^{\infty} C_n L_n^k(x^3) \exp(-\lambda nt). \tag{39}$$

The coefficients C_n are obtained from the initial distribution function $f(x, 0)$ for incident particles and the orthogonality condition as

$$C_n = \frac{n!}{(n+k)!} \int_0^{\infty} f(x,0) L_n^x(x^3)\, dx. \tag{40}$$

4. Discussion

If the distribution function for test particles has the shape of a δ-function *i.e.*, $f(x, 0) = \delta(x - x_0)$, then the expansion coefficient C_n in eq. (40) will take the form

$$C_n = \frac{n!}{(n+5/3)!} \int_0^{\infty} \delta(x - x_0) L_n^{5/3}\, dx. \tag{41}$$

The distribution function of test particles at any instant after going back from variable x to v, is

$$f_t(v,t) = v^5 e^{-v^3} \sum_{n=0}^{\infty} \frac{n!}{(n+5/3)!} L_n^{5/3}(v^3) L_n^{5/3}(v^3) \exp(-\lambda_m t). \tag{42}$$

This gives the desired distribution function of relaxed test particles at any instant. It shows in fact, the evolution of the initial δ-function shaped distribution function of test particles.

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